FAN-OUT FILTERS

by

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INTRODUCTION

In the operation of most filters only the impedance characteristics in the pass band of the particular filter in question need be given much consideration. However, when the inputs of filters whose pass bands are not the same are paralleled the equivalent impedance of the resultant network needs to be as nearly a constant resistance as possible throughout the pass bands of any of the individual filters.

This paper reviews previous work on the operation of filters in fan fashion (input sides in parallel) which starts with the characteristic impedance improvement of a single filter and then uses similar techniques adapted to filters which have their inputs in parallel. It then presents methods for obtaining different degrees of approximation to a constant resistance for the input impedance to the filters in question. This is done for both complementary filters and for low-pass filters with different bandwidths.

The interactance of a filter is defined in order to have a comparison measure between fan-out filters which have input impedances approximating a constant resistance with different degrees of approximation.

Other filter configurations are discussed which have maximum power transfer both at the common input to the filters and at the individual filter outputs. The additional resistors in these networks cause large insertion loss and this makes the filters undesirable for most applications.

In order to simplify calculations, both the cut-off frequencies and the impedance levels of most of the filters discussed in this paper have been normalized to 1.

PREVIOUS WORK

O. J. Zobel's Patent

As early as 1920, O. J. Zobel (8) gave a description of a method for paralleling the inputs of two complementary constant-K filters. Complementary filters were defined as filters which have pass bands and attenuation bands which are approximately the opposites of each other. An example of such filters would be a high-pass and a low-pass filter with the same cut-off frequency.

His method started from what he called impedance improvement of a single filter. This consisted of adding either x-series terminations or x-shunt terminations, of which the x-series termination will be discussed here. The impedance improvement method was based upon the assumption that the filter whose impedance was to be improved was terminated in its characteristic impedance. Then by adding an x-series termination, consisting of a series element which was 0.809 times that of the full series element of the original filter, and adding in shunt, a shunt annulling element which consisted of $0.5Z_1$ in series with $3.236Z_2$, (where Z_1 is the full series arm of the original filter and Z_2 is the full shunt arm of the original filter), it was found that the input impedance of the filter was reasonably flat throughout most of the pass band of the filter. These values for the added elements were found by plotting the input impedance of the corresponding circuit for different x-series terminations and noting which values would give approximately a flat input impedance throughout most of the pass band of the filter. Also, in practice these values were found desirable because of a similarity in the elements of both shunt and series annulling networks. Series annulling networks are employed with the use of x-shunt terminations.

By following this procedure for both a high- and a low-pass filter Zobel noted that the annulling network for the low-pass filter was a shunt branch which consisted of a capacitor in series with an inductance. Thus the x-series termination of the high-pass filter could be considered as the capacitor and the rest of the high-pass filter could serve as the inductance if the inputs of the two filters were paralleled. In other words, the high-pass filter would act as the reactance annulling circuit for the low-pass filter. In the same manner it was noted that the low-pass filter would serve as the reactance annulling network for the high-pass filter throughout most of the latter's pass band.

Zobel also states that the load side of each of the filters in question should be terminated in such a way as to cause the impedance of each filter, as viewed from that end, to be approximately a constant resistance over the range of transmission of that filter.

W. H. Bode's Paper

W. H. Bode (1) expanded upon Zobel's fundamental idea of impedance correcting networks for a single filter. His approach was to consider a network similar to Fig. 1. He first assumed the impedance to be corrected was the characteristic impedance of a constant-K "T" section or "Π" section. Then writing the expression for the input admittance, and neglecting the susceptance annulling network, he determined the values of the various added elements which would make the conductance an approximation to a constant. After having determined the values to use for the conductance, or resistance controlling network, he added a susceptance annulling network which would approximately cancel the susceptance throughout the pass band of the filter.

Using the case in which only one element is added to the conductance controlling network of a "T" filter (see Fig. 2) as an example, Bode obtained the following results. The expression for the input admittance is

$$Y_{in} = G + jB = \frac{\sqrt{(1 - x^2)} - ja_1x}{Z_o\left[1 - (1 - a^2)x^2\right]} \tag{1}$$

where

$$x = \frac{Z_1}{j2Z_o} \tag{2}$$

The conductance is

$$G = \frac{\sqrt{(1 - x^2)}}{Z_o\left[1 - (1 - a^2)x^2\right]} \tag{3}$$

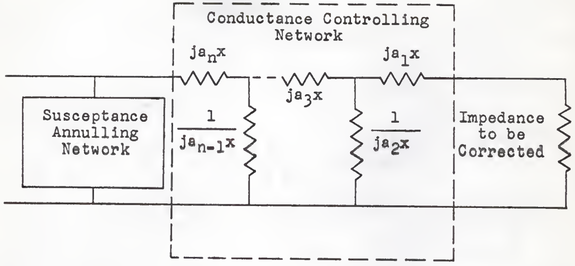

Fig. 1. Generalized schematic of impedance correcting network.

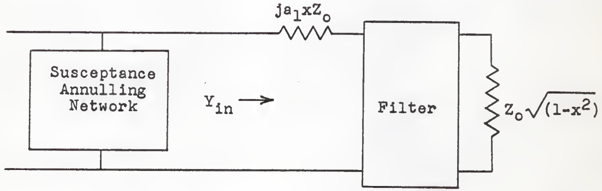

Fig. 2. Configuration of one-branch termination.

and the susceptance is

$$B = \frac{-a_1 x}{Z_o\left[1 - (1 - a^2)x^2\right]} \tag{4}$$

This susceptance may be annulled by a two-terminal impedance in parallel with the rest of the network which has the value of 1/B but has the opposite sign. Therefore the susceptance annulling network is an impedance

$$\frac{Z_o\left[1 - (1 - a^2)x^2\right]}{ja_1 x} = \frac{Z_o}{ja_1 x} - \frac{jxZ_o(a^2 - 1)}{a_1} \tag{5}$$

This is recognized as a series combination of two elements by using equation (2) and also for a constant-K filter

$$Z_1 Z_2 = Z_o^{\,2} \tag{6}$$

Equation (5) then becomes

$$\frac{Z_o}{ja_1 x} - \frac{jxZ_o(a^2 - 1)}{a_1} = \frac{2}{a_1} Z_2 + \frac{1 - a_1^{\,2}}{2a_1} Z_1 \tag{7}$$

The choice for the value of a_1 is based upon the fact that it is desired to make the conductance approximately equal to a constant. Therefore the denominator of equation (3) is made to approximate $\sqrt{(1 - x^2)}$. Bode suggested the use of the binomial theorem to expand $\sqrt{(1 - x^2)}$ and then set as many like powers of x equal as possible in the numerator and denominator polynomials of equation (3). The expansion of

$$\sqrt{(1 - x^2)} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 + \cdots\cdots \tag{8}$$

Substituting equation (8) into equation (3) gives

$$G = \frac{1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \ldots .}{Z_o\left[1 - (1 - a^2)x^2\right]} \tag{9}$$

Setting like powers of x equal gives

$$-\frac{1}{2} = (a^2 - 1) \tag{10}$$

or

$$a_1 = 0.707 \tag{11}$$

This approach yields the same results for the terminating section as does m-derivation if a value of m = 0.707 is used. However, if more than one added reactance is used in the resistance or conductance controlling network it differs considerably from m-derivation.

Bode also suggests that the best approximation (with "least square" deviation) is obtained by expressing the $\sqrt{(1 - x^2)}$ as a sum of Legendre coefficients instead of the binomial expansion used in the preceding example.

E. A. Guillemin's Book

E. A. Guillemin (2) extended the method used by Zobel for paralleling the inputs of filters to the class called potentially complementary filters. Potentially complementary filters are those which have different pass and attenuation bands but do not have the same cut-off frequencies.

Guillemin started by studying the characteristic impedance of a uniform ladder structure of the constant-K type with fractional series termination as suggested by Zobel.

He wrote the expression for the input impedance (see Fig. 3) which was

$$Z_{in} = R\sqrt{(1 - x^2)} + jaxR \tag{12}$$

where

$$Z_1 = 2jxR \tag{13}$$

and

$$Z_2 = \frac{R}{2jx} \tag{14}$$

For further consideration he studied the corresponding admittance function

$$Y_{in} = \frac{1}{Z_{in}} = G + jB \tag{15}$$

or in particular, the real and imaginary parts G and B.

$$R(G + jB) = \frac{1}{\sqrt{(1 - x^2)} + jax} \tag{16}$$

Since the factor $\sqrt{(1 - x^2)}$ may be either real or imaginary depending upon whether a pass band or an attenuation band is considered, Guillemin studied the admittance function separately in these ranges.

Thus for the range $-\infty < x \leq 1$

$$RG = 0 \tag{17}$$

and

$$RB = \frac{1}{\sqrt{(x^2 - 1)} - ax} \tag{18}$$

This followed from the fact that the minus sign of the radical was used to enable the $\sqrt{(x^2 - 1)}$ to represent a reactance with a positive slope which was necessary for realization. Also for the range $-1 \leq x \leq 1$

$$RG = \frac{\sqrt{(1 - x^2)}}{1 - (1 - a^2)x^2} \tag{19}$$

and
$$RB = \frac{-ax}{1 - (1 - a^2)x^2} \tag{20}$$

and for the range $1 \leq x < \infty$

$$RG = 0 \tag{21}$$

and
$$RB = \frac{-1}{\sqrt{(x^2 - 1)} + ax} \tag{22}$$

Then he applied these equations to both the high-pass and the low-pass filters. For the low-pass case $x = \omega/\omega_c$. The application of the low pass case to equation (9) and equation (11) gave

$$RB = \frac{-\left(\frac{\omega}{\omega_c}\right)}{1 - (1 - a^2)\left(\frac{\omega}{\omega_c}\right)^2} \; ; \; 0 \leq \frac{\omega}{\omega_c} \leq 1 \tag{23}$$

and
$$RB = \frac{-1}{\sqrt{\left(\frac{\omega}{\omega_c}\right)^2 - 1} + a\left(\frac{\omega}{\omega_c}\right)} \; ; \; 1 \leq \frac{\omega}{\omega_c} < \infty \tag{24}$$

and for the high-pass case $x = -\omega_c/\omega$ which gives

$$RB = \frac{1}{\sqrt{\left(\frac{\omega_c}{\omega}\right)^2 - 1} + a\left(\frac{\omega_c}{\omega}\right)} \; ; \; 0 \leq \frac{\omega}{\omega_c} \leq 1 \tag{25}$$

$$RB = \frac{a\left(\frac{\omega_c}{\omega}\right)}{1 - (1 - a^2)\left(\frac{\omega_c}{\omega}\right)} \; ; \; 1 \leq \frac{\omega}{\omega_c} < \infty \tag{26}$$

The plots of these equations showed that when the two filter

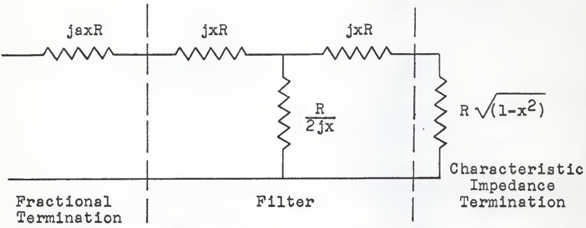

Fig. 3. The properly terminated constant-K "T" section with a fractional series branch on the input side.

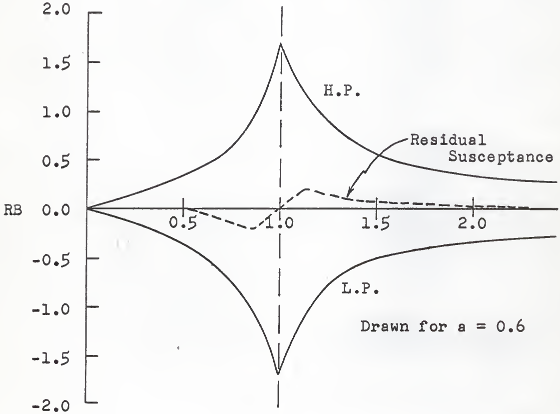

Fig. 4. Susceptance characteristics for a pair of complementary filters with fractional terminations and identical cut-off frequencies.

inputs were paralleled, each filter would serve as an approximate annulling circuit for the other provided the cut-off frequencies were identical (see Fig. 4). However, he noted that the susceptance characteristics did not cancel each other very well if the cut-off frequencies were not the same (see Fig. 5). He found it necessary in this case to place another susceptance in parallel with the input sides of the filters. This additional network was a resonant circuit with a resonant frequency of

$$\omega_a = \frac{1}{\sqrt{(L_a C_a)}} \tag{27}$$

and the susceptance of the network when multiplied by R was

$$RB_a = \frac{R}{L_a(\omega_a{}^2 - \omega^2)} \tag{28}$$

By properly choosing the resonant frequency ω_a, and the values of L_a and C_a, Guillemin found that a reasonably constant input resistance could be maintained in the pass band of each of the filters. He suggested the use of the geometric mean of the two filter cut-off frequencies for ω_a. The values of L_a and C_a are then determined by the frequencies at which the susceptances are to be completely corrected.

E. L. Norton's Paper

E. L. Norton (5) used a somewhat different approach to the problem of paralleling the inputs of complementary filters. He determined what the form of the equivalent input impedance needs to be in order to be a constant resistance, and then synthesized

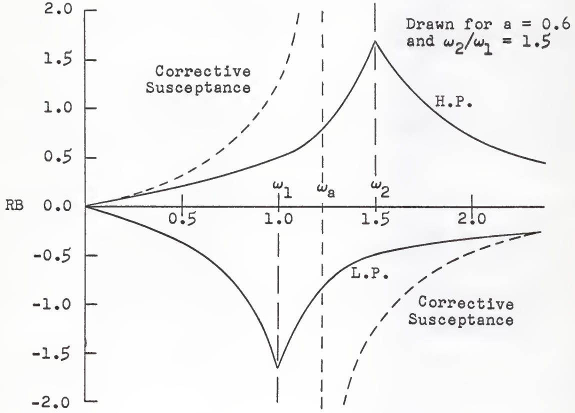

Fig. 5. Susceptance characteristics and corrective susceptance for a pair of potentially complementary filters with fractional series terminations but different cut-off frequencies.

a pair of complementary filters which had this form of input impedance. This gave the desired input impedance; however, the calculation required to synthesize a complex filter of this type is rather long since it involves the solution of several simultaneous nonlinear equations.

DEFINITION OF INTERACTANCE

In order to have some means of comparing the power available at the input terminals of a filter, or a group of filters whose inputs are connected in parallel, with that which would be available if the networks were purely resistive, the term "interactance" is used. Interactance is defined as

$$\Lambda = \frac{P_f}{P_r} \tag{29}$$

where Λ is the interactance of the filter, P_f is the power dissipated in the source resistance when the filter or filters are connected, and P_r is the power dissipated in the source resistance when a pure resistance replaces each filter in the circuit. These powers, P_f and P_r, are measured with an additional voltage put in each parallel filter path (see Fig. 6) such that when the filter is replaced by a pure resistance the voltage across the source resistance is equal to the original source voltage. Figure 7 shows the filters replaced by one-ohm (which is the pseudo-characteristic impedance of the filters) resistors. Using Millman's theorem, the voltage e_{or} in Fig. 7 is

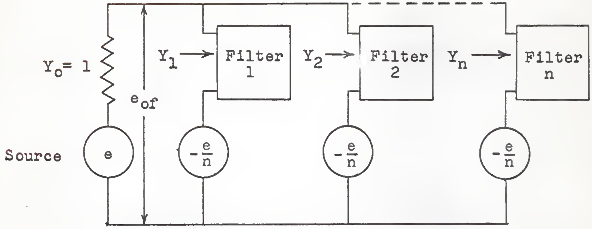

Fig. 6. The circuit arrangement for measurement of the interactance of a group of filters whose inputs are in parallel.

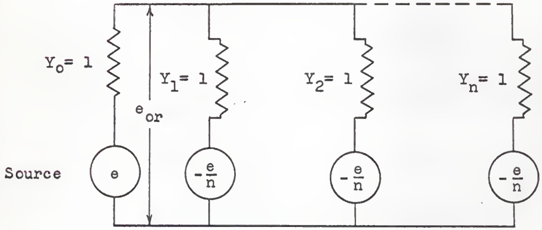

Fig. 7. The circuit arrangement for measurement of the interactance with the filters replaced by one-ohm resistors.

$$e_{or} = \frac{1(e) + n\left(\frac{-e}{n}\right)}{1 + n} \tag{30}$$

where n is the number of parallel paths and is always a positive integer. Solving equation (30) it is found that

$$e_{or} = \frac{e(1 - 1)}{1 + n} = 0 \tag{31}$$

Therefore the voltage across the source resistance (e_r) is

$$e_r = e - e_{or} = e \tag{32}$$

where e is the source voltage. The power, P_r, dissipated in the source resistance under these conditions is (assuming e is a peak value)

$$P_r = \frac{1}{2}\frac{|e|^2}{R} \tag{33}$$

If R is one ohm, equation (33) reduces to

$$P_r = \frac{1}{2}|e|^2 \tag{34}$$

The calculation of P_f follows a similar procedure except the one-ohm load resistors are replaced by the admittance Y_n of each filter as in Fig. 6.

$$e_{of} = \frac{Y_o(e) + Y_1\left(\frac{-e}{n}\right) + Y_2\left(\frac{-e}{n}\right) + \cdots\cdot + Y_n\left(\frac{-e}{n}\right)}{Y_o + Y_1 + Y_2 + \cdots\cdot + Y_n} \tag{35}$$

The voltage across the source resistance for this case is

$$e_r = e - e_{of} \tag{36}$$

Substituting equation (35) into equation (36) yields

$$e_r = e \left[\frac{\frac{n+1}{n}(Y_1 + Y_2 + \cdots + Y_n)}{Y_o + Y_1 + Y_2 + \cdots + Y_n} \right] \tag{37}$$

The power, P_f, in this case is

$$P_f = \frac{1}{2}|e|^2 Y_o \left| \frac{\frac{n+1}{n}(Y_1 + Y_2 + \cdots + Y_n)}{Y_o + Y_1 + Y_2 + \cdots + Y_n} \right|^2 \tag{38}$$

Substituting equations (34) and (38) into equation (29), the interactance is found to be

$$\Lambda = Y_o \left| \frac{\frac{n+1}{n}(Y_1 + Y_2 + \cdots + Y_n)}{Y_o + Y_1 + Y_2 + \cdots + Y_n} \right|^2 \tag{39}$$

in terms of the input admittances of the individual filters in the circuit.

Calculating the interactance in the preceding manner, it is noted that the plot of ideal interactance versus frequency is a straight line. The value for the ideal interactance is one. However, as in the case of complementary filters, the admittance of one filter may be zero, so that the added voltage in that branch adds no power to the circuit. Therefore the best interactance plot that may be obtained for this case is a straight line of some value less than one. Thus it is necessary to consider each case separately in order to determine the best value of interactance obtainable, although the straight line applies to all cases.

INTERACTANCE OF INDIVIDUAL FILTERS

Constant-K "T" Filter

Considered individually, the symmetrical "T" or "Π" constant-K filter has an input impedance which approximates a pure resistance of K ohms. This is seen by examining the input impedance of a low-pass "T" constant-K filter (see Fig. 8). The expression for the input impedance (K normalized to one ohm) is

$$Z_{in} = \frac{1 + 2s + 2\lambda s^2 + 2\,(2 - \lambda)s^3}{1 + 2s + 2(2 - \lambda)s^2} \tag{40}$$

If λ is chosen such that the maximum number of coefficients of corresponding powers of s in the numerator and denominator are equal, the filter is symmetrical and the input impedance is

$$Z_{in} = \frac{1 + 2s + 2s^2 + 2s^3}{1 + 2s + 2s^2} \tag{41}$$

This is King's (4) approximation to a constant for small values of s -- as many as possible of the corresponding powers of s in the numerator and denominator polynomials are equal.

The interactance of this low-pass filter is calculated by use of equation (39) and is

$$\Lambda = \left| \frac{1 + 2s + 2s^2}{1 + 2s + 2s^2 + s^3} \right|^2_{s\,=\,j\omega} \tag{42}$$

Evaluating Λ for $s = j\omega$, the following result is obtained.

$$\Lambda = \frac{1 + 4\omega^4}{1 + \omega^6} \tag{43}$$

Figure 9 is a plot of Λ versus frequency. It is noted that the

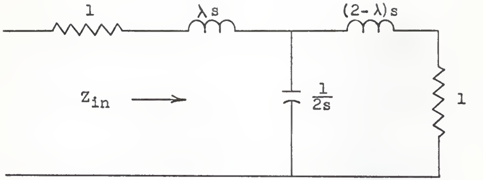

Fig. 8. Low-pass constant-K filter terminated in a one-ohm resistor.

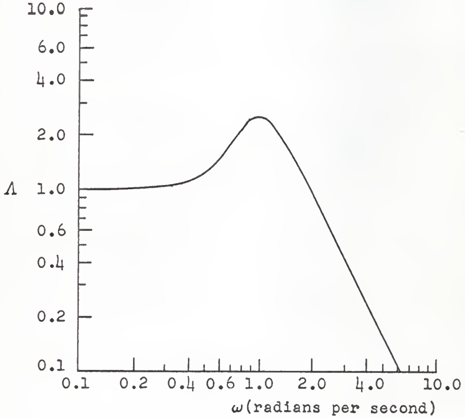

Fig. 9. Interactance of the low-pass constant-K filter shown in Fig. 8 with $\lambda = 1$.

curve starts off with a flat response but then changes value rapidly about the middle of the pass band. In order to obtain an interactance plot which is flat throughout more of the pass band it is necessary to use a filter which has an input impedance which more closely approximates a constant, such as with m-derived filters or filters which have some other type of impedance improvement.

Figure 10 shows the interactance for a high-pass constant-K filter with the same cut-off frequency as the low-pass filter.

M-Derived Filter

The m-derivation of a constant-K filter, which is included in most network textbooks, is one means of improving the impedance characteristics throughout the pass band of the filter. It also makes it possible to choose one frequency of infinite attenuation which depends upon the value of m that is chosen. This means that it not only has a flatter impedance characteristic in the pass band but also has a sharper cut-off at the start of the attenuation band.

The interactance of a constant-K filter which has m-derived terminating half-sections is shown in Fig. 11. This curve is plotted for $m = 0.6$, a value that gives good filter characteristics. A value of 0.707 for m would give a somewhat better characteristic impedance.

Comparing Fig. 11 with Fig. 9 shows that the m-derived filter has a much better interactance plot throughout the pass

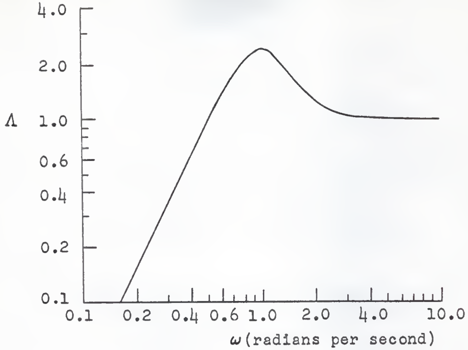

Fig. 10. Interactance of a high-pass constant-K filter which is the complement of the filter shown in Fig. 8.

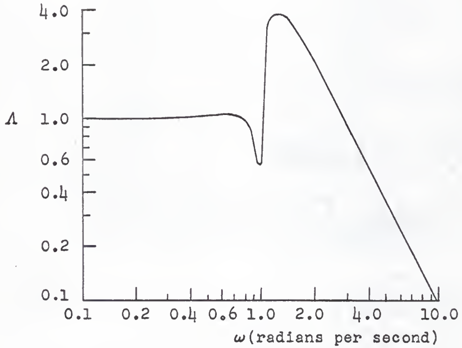

Fig. 11. Interactance of a low-pass m-derived filter with $m = 0.6$.

band than the constant-K filter. However, it is noted that the m-derived filter interactance has a high peak just outside its pass band at a frequency which corresponds to the frequency of infinite attenuation. This, as would be expected, causes trouble when a high- and a low-pass m-derived filter are connected in fan fashion.

INTERACTANCE OF COMPLEMENTARY FAN-OUT FILTERS

Fan-Out Constant-K Filters

If two complementary constant-K filters are connected in fan fashion, the interactance of the combined network may be calculated by use of equation (39).

$$\Lambda = \left|\frac{\frac{3}{2}(1 + 4s + 8s^2 + 9s^3 + 8s^4 + 4s^5 + s^6)}{2(1 + 3.5s + 6.5s^2 + 7.75s^3 + 6.5s^4 + 3.5s^5 + s^6)}\right|^2_{s = j\omega} \qquad (44)$$

Evaluating Λ for various values of ω, the curve of Fig. 12 is obtained. This curve is not flat throughout as much of the pass band of each of the filters as is the curve for the interactance of each filter by itself (see Figs. 9 and 10). However, the peak at the cut-off frequency is not as high as for the individual filters.

That the equivalent input impedance for the fan-out constant-K filters is a ratio of polynomials of higher degree of s than for each filter by itself suggests that more of the coefficients of corresponding powers of s in the numerator and denominator polynomials could be made equal and the plot of interactance

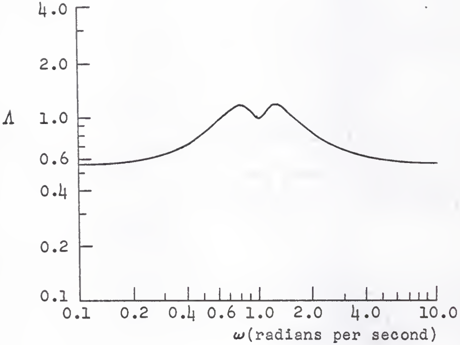

Fig. 12. Interactance of the high-pass and low-pass constant-K filters when their inputs are in parallel.

could be a much flatter curve. One method of accomplishing this is to write the expression for the equivalent input impedance with a variable coefficient for the reactance nearest the input terminals in each filter (Fig. 13). This gives an equivalent input impedance of

$$Z_{in} = \frac{A_o + A_1 s + A_2 s^2 + A_3 s^3 + A_4 s^4 + A_5 s^5 + A_6 s^6}{B_o + B_1 s + B_2 s^2 + B_3 s^3 + B_4 s^4 + B_5 s^5 + B_6 s^6} \qquad (45)$$

where

$$\begin{aligned} A_o &= 2k_1 & B_o &= 2k_1 \\ A_1 &= 2k_1(k + 2) & B_1 &= 6k_1 + 2 \\ A_2 &= 6kk_1 + 3k + 1 & B_2 &= 9k_1 + 2k + 5 \\ A_3 &= 10kk_1 + 2k + 1 & B_3 &= 6k_1 + 6k + 6 \\ A_4 &= 6kk_1 + 3k + 1 & B_4 &= 2k_1 + 9k + 5 \\ A_5 &= 2k(k_1 + 2) & B_5 &= 6k + 2 \\ A_6 &= 2k & B_6 &= 2k \end{aligned} \qquad (46)$$

The approach is now to choose k and k_1 so that the input impedance is as close an approximation to one, in both pass bands, as can be obtained. Since the coefficients of the zero and sixth powers of s will be equal with any value of k and k_1, the coefficients of the first power of s and the fifth power of s are examined. The coefficient of s in the denominator is set equal to the numerator s coefficient. The same procedure is followed for the coefficients of the fifth power of s. From equations (44) and (45) this gives

$$2k_1(k + 2) = (6k_1 + 2) \qquad (47)$$

and

$$2k(k_1 + 2) = (6k + 2) \qquad (48)$$

Solving equations (47) and (48) simultaneously for k and k_1

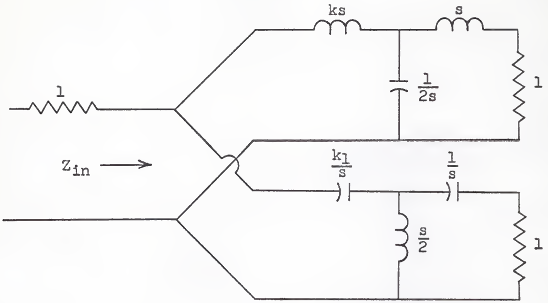

Fig. 13. Circuit configuration used in the first step of improving the interactance of complementary constant-K filters.

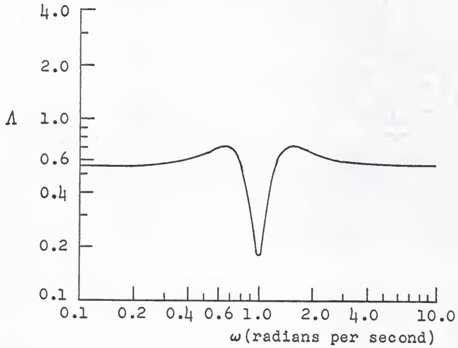

Fig. 14. Interactance of the complementary filters of Fig. 13 with $k = k_1 = 1.618$.

gives the following result

$$k = k_1 = 1.618 \tag{49}$$

Substituting these values into equations (45) and (46), and solving for the interactance by use of equation (39), the curve of Fig. 14 is obtained. This gives a much closer approximation of a straight line for the interactance than when the two filters were merely paralleled without changing any element values. It is also noted that this curve is flatter throughout more of the pass band regions than are the individual filter interactance curves.

If the magnitude of the voltage transfer function of each of the filters is plotted versus frequency for the filters by themselves and when in parallel, it is seen that there is a big improvement when their inputs are paralleled and the first series impedance changed to 1.618 times its original value (see Figs. 15 and 16). In fact, it approaches the flatness and sharpness of the cut-off of the transfer characteristic of a single m-derived filter.

This value of $k = 1.618$ agrees very closely with the value Zobel (8) used in his x-terminations which were determined from practical consideration and from calculations considering only perfect terminations for the filters in question.

Going one step further and writing the expression for the input impedance with both the series and shunt element nearest the input terminals of the filters having a variable coefficient (see Fig. 17), it is possible to get another approximation to a

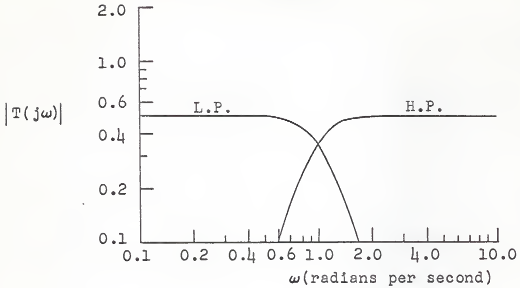

Fig. 15. Voltage transfer functions of complementary low-pass and high-pass constant-K filters when each is used alone.

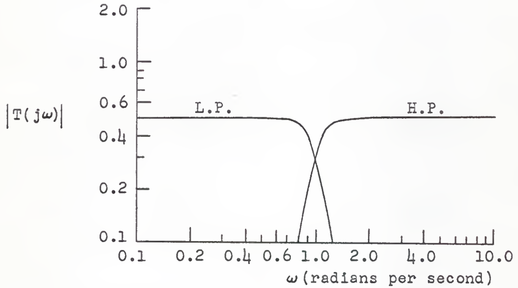

Fig. 16. Voltage transfer function for complementary high-pass and low-pass fan-out filters which have the interactance characteristics shown in Fig. 14.

constant for the input impedance. This allows the coefficients of the s^0, s^1, s^2, s^4, s^5, and s^6 terms of the numerator polynomial to be set equal to the coefficients of the corresponding terms in the denominator. Also, by noting that for complementary filters the coefficients of s of the numerator and denominator have the relationship $A_o = A_n$, $A_1 = A_{n-1}$, $A_2 = A_{n-2}$, ... , and $B_o = B_n$, $B_1 = B_{n-1}$, $B_2 = B_{n-2}$, ... , the number of equations necessary for this case is reduced to two. Carrying out the preceding calculations the following simultaneous equations are obtained.

$$kk_1 = k^2 + k - 1 \tag{50}$$

and

$$2kk_1{}^2 + kk_1 + 2k_1 + k = k^2k_1{}^2 + k^2k_1 + 2k \tag{51}$$

Solving these equations for the roots which yield realizable reactances gives

$$k = 1.73 \tag{52}$$

and

$$k_1 = 2.15 \tag{53}$$

Using these values for the calculation of the interactance, the plot of Fig. 18 is obtained. This second approximation gives only a slight increase in the flatness of the interactance curve at the low and high frequencies, and its characteristics near the cut-off frequency are not as desirable as in the previous case. Also, the voltage transfer function does not have as sharp a cut-off.

If this procedure is carried still further and all of the reactive elements are allowed to have variable magnitudes in the calculation of the input impedance, and setting the coefficients of corresponding powers of s in the numerator and denominator

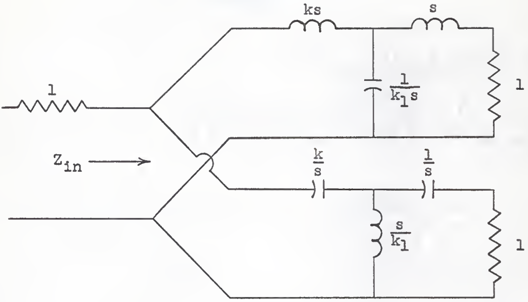

Fig. 17. Circuit configuration used for the second step of improving the interactance of complementary constant-K filters.

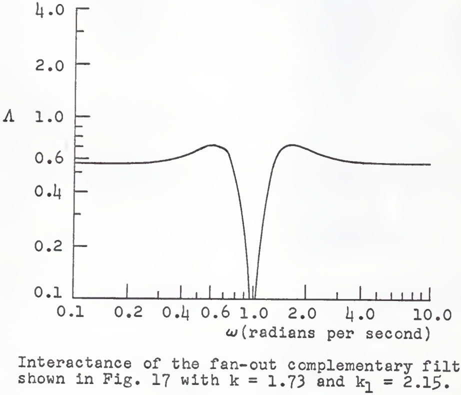

Fig. 18. Interactance of the fan-out complementary filters shown in Fig. 17 with $k = 1.73$ and $k_1 = 2.15$.

equal, and solving these equations simultaneously, the values which make the input impedance a constant resistance are obtained. This procedure requires the solution of three simultaneous nonlinear equations for this simple case, and becomes very difficult for more complex filters. Carrying out the preceding calculations gives values of

$$k = \frac{3}{2} \tag{54}$$

$$k_1 = \frac{4}{3} \tag{55}$$

$$k_2 = \frac{1}{2} \tag{56}$$

Although the interactance is a straight line for this case, the voltage transfer function is not as flat as in the previous case. This means that some compromise will have to be made in a practical situation between the input impedance and the filter characteristics required.

The circuit obtained for this constant resistance case resembles the circuit which Norton (5) used in his constant resistance synthesis approach.

Fan-Out M-Derived Filters

Figure 19 shows the interactance of fan-out high-pass and low-pass m-derived filters. This curve shows that the characteristics of these filters are not even as good as the characteristics of the fan-out constant-K filters discussed previously.

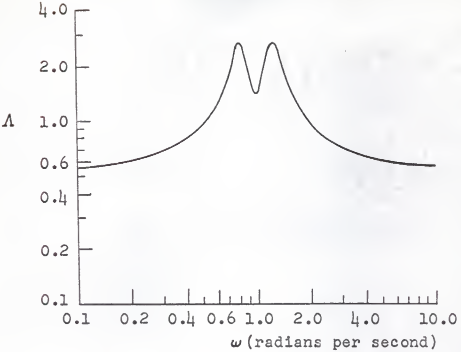

Fig. 19. Interactance of fan-out low-pass and high-pass complementary m-derived filters.

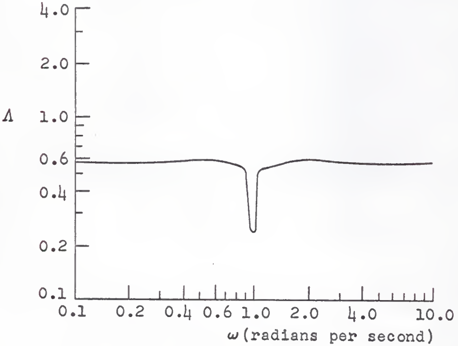

Fig. 20. Interactance of fan-out low-pass and high-pass complementary degenerate m-derived filters.

Since the shunt resonant branch nearest the input of the m-derived filter is used to give the frequency of infinite attenuation, or in this case an impedance zero, it is now omitted from the circuit. This will be referred to as a degenerate case of m-derivation. When the interactance of the high-pass and the low-pass degenerate m-derived filters connected in fan fashion is plotted, the curve of Fig. 20 is obtained. A value of $m = 0.6$ was used again for these calculations. This plot shows a marked improvement in the interactance over all other cases considered thus far except the final case of the constant-K filters. Also, the voltage transfer function for the degenerate m-derived filters in parallel is flat over most of the pass bands of the filters and has sharp cut-off characteristics. In fact, the cut-off is even sharper for this case than for a regular m-derived filter by itself.

If a procedure similar to that used for finding the best value of the series impedance element nearest the input in each of the parallel constant-K filters is used for the degenerate m-derived case, it is found that a value of 1.618 should be used instead of 1.6. This is the same value that was found to give the best results for constant-K filters.

As in the complementary constant-K filter examples, the process of setting equal the coefficients of corresponding powers of s in the numerator and denominator polynomials of the input impedance function could be carried to such an extent that a network would result which would resemble the complementary degenerate m-derived filters in parallel and have a constant resistance for the input impedance. However, as with the constant-K filters, the voltage transfer function is not as good for this case as it is for the degenerate m-derived case.

Fan-Out Filters Derived from Self-Dual Filters

The input impedance of a low-pass symmetrical constant-K "T" filter as shown in Fig. 8 is

$$Z = \frac{1 + 2s + 2s^2 + 2s^3}{1 + 2s + 2s^2} \tag{57}$$

If the impedance level is doubled, the input impedance is

$$Z_{in} = \frac{2 + 4s + 4s^2 + 4s^3}{1 + 2s + 2s^2} \tag{58}$$

which may also be written as

$$Z_{in} = 2 + \frac{4s^3}{1 + 2s + 2s^2} = 2 + \epsilon \tag{59}$$

The input impedance of a filter which is the dual of Fig. 8 with the impedance level doubled is

$$Z_{in} = \frac{2 + 4s + 4s^2}{1 + 2s + 2s^2 + 2s^3} = 2 - \frac{2s^3}{1 + 2s + 2s^2 + 2s^3} \cong 2 - \epsilon \tag{60}$$

If these two filters are connected so that their inputs are in parallel, the input impedance is a closer approximation to a resistance for small values of ϵ.

$$\frac{(2 + \epsilon)(2 - \epsilon)}{2 + \epsilon + 2 - \epsilon} = 1 - \frac{\epsilon^2}{4} \cong 1 \tag{61}$$

Since the voltage out of each of the filters is the same the output terminals may also be connected in parallel. The resulting network is the self-dual network shown in Fig. 21. Reduction of this network to an equivalent ladder structure yields the low-pass filter of Fig. 22.

A high-pass filter with an improved impedance characteristic may be obtained in a similar manner. The fan-out operation of these two filters is not very satisfactory since the low-pass filter has an impedance zero at $\omega = \infty$ and the high-pass filter has an impedance zero at $\omega = 0$. However, if the parallel branch nearest the input terminals of each of the filters is removed, as in the fan-out operation of m-derived filters, the input impedance characteristics are greatly improved. Figure 23 shows the interactance of these networks which is an improvement over the constant-K filter interactance. Also, the voltage transfer functions (see Fig. 24) of these filters connected in fan fashion are an improvement over the transfer functions of the constant-K filters. In fact, they have similar cut-off characteristics to the transfer functions of the degenerate m-derived filters.

The interactance of the complementary fan-out filters derived from the parallel connection of dual filters may be improved even more by the method used on the constant-K and the m-derived filters. This involves the changing of the first series element in each filter to obtain the coefficient of s^1 in the numerator which will equal the coefficient of s^1 in the

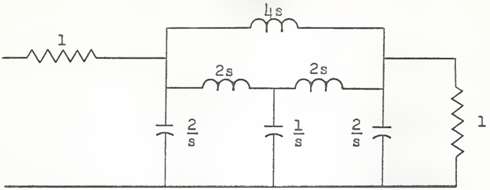

Fig. 21. The self-dual network resulting from the paralleling of dual low-pass constant-K filters.

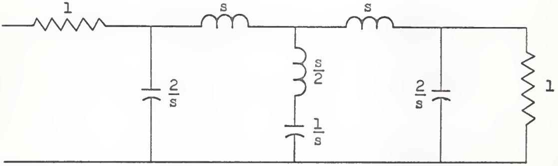

Fig. 22. Ladder equivalent network of Fig. 21.

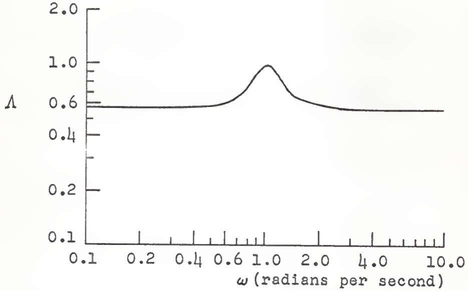

Fig. 23. Interactance of fan-out complementary filters derived from self-dual filters.

denominator of the input impedance function. Carrying out these calculations gives a value of 1.167s instead of s as shown in Fig. 22. Figure 25 shows the improved interactance curve. As was the case with the constant-K and m-derived filters, this process of changing element values could be carried to such an extent as to yield complementary filters with a constant resistance input impedance.

INTERACTANCE OF LOW-PASS FAN-OUT FILTERS

Fan-Out Filters with Identical Pass Bands

If the inputs of two filters whose pass bands are identical are paralleled the problem of obtaining a flat interactance curve is very similar to that associated with a single filter. Impedance improvement of each of the individual filters results in an overall impedance improvement of the parallel combination. One additional method may be employed in this case. If one of the filters to be paralleled is made to be the dual of the other the resultant input impedance of the parallel combination is much closer to a constant resistance in the pass band. This is seen by writing the expression for the input impedance of a low-pass constant-K filter and its dual as was done in the section on filters derived from self-dual filters. The only difference is that the output terminals are not connected in parallel in this case. The application of impedance improvement methods upon these dual filters results in a good approximation of a constant resistance in the pass band. The impedance level of

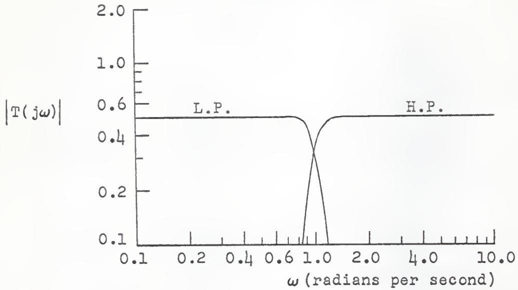

Fig. 24. Voltage transfer functions of fan-out complementary filters derived from self-dual filters.

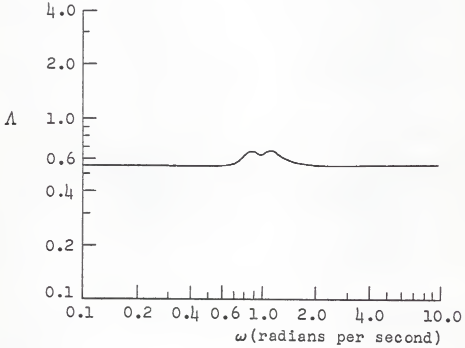

Fig. 25. Interactance of improved fan-out filters derived from self-dual filters.

the filters should also be doubled to get closer to maximum power transfer in this case.

Fan-Out Filters with Different Bandwidths

Since two low-pass filters of different bandwidths do not act as reactance annulling circuits for each other, it is necesary to find a method for making their impedance approximately constant. The case in which the bandwidth of one low-pass filter is twice the bandwidth of the low-pass filter with which it is to be paralleled will be considered here.

In the constant-K and m-derived filters considered previously in this paper, the characteristic impedance becomes purely reactive at the cut-off frequency. In the case under consideration here one filter has a resistive characteristic impedance and one a reactive characteristic impedance in the region where their pass bands do not overlap. However, by connecting two sections of the filter with the higher cut-off frequency in cascade, a filter of half the bandwidth is obtained, although its characteristic impedance remains resistive throughout the bandwidth of the original filter. If a filter made up of these two sections is connected in fan fashion with the original one-section filter the resulting equivalent input impedance may be made to approximate a constant over the wider pass band of either of the filters. Figure 26 shows an example of the preceding network using simple constant-K filters. The interactance curve for the parallel combination (see Fig. 27) resembles that of a

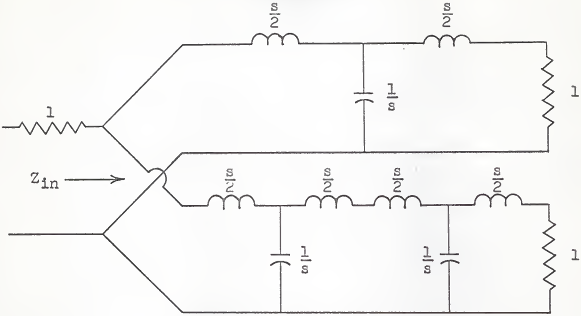

Fig. 26. Circuit configuration used to improve the interactance of two parallel low-pass filters which have different bandwidths.

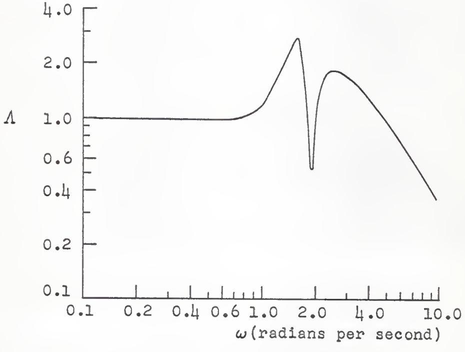

Fig. 27. Interactance of the filters shown in Fig. 26.

single low-pass filter except at the higher cut-off frequency of the two filters.

In order to improve the input impedance of the combination it is necessary to improve the impedance of each of the individual sections. This may be done with the use of self-dual networks or some other method of impedance improvement.

Although this procedure yields a reasonable approximation of a constant for the input impedance, the voltage transfer functions of the filters do not have as sharp cut-off characteristics as is sometimes desired. This is especially true for the filter with the smaller bandwidth.

If the filter with the wider bandwidth is made to be the dual of one of the sections of the two-section filter the interactance characteristics are better than for the first case discussed (see Fig. 28). Also, impedance improvement of these dual filters gives the flattest interactance response.

The procedure of cascading filter sections, which may be applied to both low- and high-pass filters, may be used to obtain filters with various ratios of bandwidths, although the number of components becomes large for large ratios.

FAN-OUT NETWORKS WITH MAXIMUM POWER TRANSFER CHARACTERISTICS

The ideal input impedance of any filter or group of filters whose inputs are in parallel is the complex conjugate of the source impedance. This is true since the condition for maximum power transfer to a load impedance is for the impedance looking

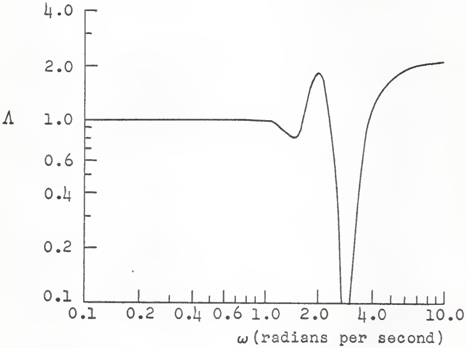

Fig. 28. Interactance of two fan-out filters of different bandwidths when the wider bandwidth filter is the dual of one of the sections of the other filter.

toward the load end to be the complex conjugate of the impedance looking toward the input end. Since the impedance of the source in the case of filters is usually a constant resistance, the impedance looking into the filters should also be a constant resistance.

Considering the ideal case, where the input impedance of each filter in a group of filters whose inputs are to be paralleled is a constant resistance, $n\zeta$, the equivalent input impedance of the parallel filters is

$$Z_{eq} = \frac{n\zeta}{n} = \zeta \tag{62}$$

where n is the number of filters in parallel. If the source impedance is also a resistance, ζ, then maximum power transfer will occur at the input terminals of the filters (see Fig. 29). However, if an individual filter of the n parallel filters is examined the impedance looking toward the load is

$$Z_L = n\zeta \tag{63}$$

The impedance looking toward the source is

$$Z_s = \zeta \tag{64}$$

Therefore, the condition for maximum power transfer to each individual load does not exist in this circuit.

If a resistance, r, is added in series with the source resistance, and also added in series with each of the loads (see Fig. 30), the conditions for maximum power transfer may be obtained for both the points a and a' and the individual filter inputs, b and b'. The impedance looking toward the

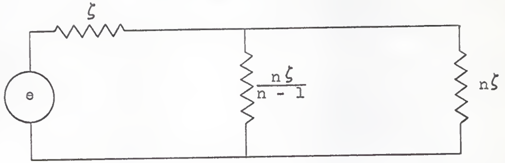

Fig. 29. Circuit diagram of n parallel resistance loads used in the maximum power transfer discussion.

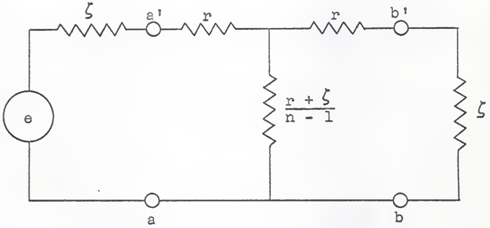

Fig. 30. Circuit diagram of parallel resistance loads with coupling resistors, r.

load terminals from points a and a' is

$$Z_L = r + \frac{r + \zeta}{n} \tag{65}$$

Setting this equal to the source resistance, ζ,

$$\zeta = r + \frac{r + \zeta}{n} \tag{66}$$

and solving for r gives

$$r = \frac{n - 1}{n + 1}\zeta \tag{67}$$

Then setting equal the impedances looking both ways at points b and b' gives

$$\zeta = r + \frac{r + \zeta}{n} \tag{68}$$

and

$$r = \frac{n - 1}{n + 1}\zeta \tag{69}$$

This value of r agrees with the value obtained in equation (67), or maximum power transfer occurs at both of these junction points for the same value of r. This type of coupling network could be used with constant resistance networks such as a bridged-T. However, even in this case, the insertion loss in the coupling resistors makes the actual power output less than for the previous case. Therefore, in most practical situations the first case would be used.

SUMMARY

Previous work which has been done on the operation of filters whose inputs are in parallel is reviewed in this paper. Most of this work developed from Zobel's patent which gave a method for impedance improvement of a single filter and a method for paralleling complementary filters. This work was based upon filters which were terminated in their characteristic impedance.

Norton's approach was a synthesis approach in that he determined the form necessary in order for the input impedance function to be a constant resistance, and then synthesized complementary filters which had this form of input impedance.

The interactance of a filter or a group of filters is defined as the ratio of the power dissipated in the source resistance to the power which would be dissipated if maximum power were transferred to the filters.

The configurations which give the best plot of interactance versus frequency do not always have the best filter characteristics. Therefore it is sometimes necessary to compromise between impedance characteristics and filter characteristics.

A method is given for finding the element values for fan-out complementary filters which make the input impedance a constant resistance or which can be used to give different degrees of approximation to a constant resistance. This is accomplished by solving for the input impedance in terms of variable reactances, and then determining the values for these reactances which will yield equal coefficients of corresponding

powers of s in the numerator and denominator polynomials.

The interactance of fan-out filters is also improved by using individual filters which have an impedance characteristic that approximates a constant resistance. Therefore impedance improvement of a single filter may be used to obtain fan-out filters with an improvement in the interactance. The use of self-dual networks to improve the impedance characteristics is demonstrated. This new method gives results which are comparable to those of m-derivation.

A procedure, which is applicable to both low- and high-pass filters, is developed for the parallel operation of two low-pass filters which have the same bandwidth, and for two which have different bandwidths. This latter case requires the resistive nature of the characteristic impedance of the filter with the lower cut-off frequency to be extended throughout the pass band of the filter with the higher cut-off frequency. This is accomplished by cascading two filter sections of the higher cut-off frequency to yield the branch which is to have the lower cut-off frequency. Additional improvement is achieved if the filter with the wider bandwidth is made to be the dual of one of the sections of the lower bandwidth filter. This makes it possible to obtain the impedance characteristics and the bandwidths desired, although the filter characteristics are not as desirable as those which may be obtained with a single filter. Therefore a compromise must be made between the filter characteristics and the impedance characteristics desired.

Although these methods yield fan-out filters which have acceptable input impedance characteristics, a general theory needs to be developed which may be applied to any type of filter.

Finally, a discussion is given of a circuit which satisfies the conditions for maximum power transfer both at the coupling network and at the individual filters. However, since the coupling network consists of resistances, the insertion loss makes the network undesirable in most cases.

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FAN-OUT FILTERS

by

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B. S., Kansas State University, 1958

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The fan-out operation of filters causes some additional problems over single filter operation. These problems are due to the variation of a normal filter's input impedance with frequency.

Most of the previous work in this field was an outgrowth of Zobel's patent on the characteristic impedance improvement of a single filter and on the parallel operation of complementary filters. This work considered the filters to be terminated in their characteristic impedance so that the impedance looking into the filter was also its characteristic impedance. The configurations which resulted were tried in practical situations until satisfactory results were obtained. Norton started with an input impedance form which was a constant resistance and then synthesized complementary filters with this impedance form.

This paper defines the interactance of a filter or a group of fan-out filters in order to have a means of comparing different configurations of filters. It is defined to be the ratio of power dissipated in the source resistance with the filters in the circuit, to the power dissipated in the same resistance when maximum power transfer conditions exist in the network. The best plot of interactance versus frequency is a straight line, the value of which depends upon the types of filters being paralleled.

A procedure for obtaining an input impedance which is a constant resistance for a particular filter configuration is to express the input impedance of the filters, which have resistive terminations, using variable terms for the reactive components. If the coefficients of s in the numerator polynomial are set equal to the coefficients of corresponding powers of s in the denominator polynomial, simultaneous equations are obtained which, when solved, will give the element values required to make the input impedance a constant resistance. By setting equal only part of the corresponding powers of s, different approximations of a constant resistance are obtained.

Individual filter impedance improvement may also be used to improve their fan-out operation. A new method of impedance improvement, the use of self-dual filters, is presented. When these filters are used in fan fashion, any reactances which yield input impedance zeros in any of the pass bands of the filters must be removed from the network.

The problem of paralleling two low-pass filters of the same cut-off frequency is similar to the problems involved with a single filter. However, if the two filters have different bandwidths a different problem arises. The resistive character of the impedance of the filter with the lower bandwidth must be extended throughout the wider bandwidth of the filter with which it is to be paralleled. This may be accomplished by cascading two sections of the wider bandwidth filter to obtain a filter with half the bandwidth and still maintain the impedance characteristics of the original filter. Then by using an impedance improvement method on each of the individual filter sections, satisfactory impedance characteristics may be obtained.

Finally, a discussion is given of a circuit configuration which gives maximum power transfer both at the input to the coupling network and at the individual filter inputs.